The Skorokhod space in functional convergence: a short introduction

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The Skorokhod space $\mathbb{D} = \mathbb{D}([0,1] : \mathbb{R}^1)$ consists of functions $x : [0,1] \to \mathbb{R}^1$ which admit limit $x(t-)$ from the left at each point $t \in (0,1]$ and limit $x(t+)$ from the right at each point $t \in [0,1)$. After regularization by taking the right-continuous version, such functions are often called "càdlàg", what is an acronym taken from French. The Skorokhod space provides a natural and convenient formalizm for description of trajectories of stochastic processes admitting jumps, in particular: trajectories of Poisson process, Lévy processes, martingales and semimartingales, empirical distribution functions, trajectories of discretizations of stochastic processes, etc.

The supremum norm converts the Skorokhod space into a *nonseparable* Banach space, what is always disadvantageous in probability theory. Moreover, for discontinuous elements of $\mathbb{D}$, simple and natural approximation procedures like discretizations are convergent uniformly only in exceptional cases. Therefore the four metric separable topologies J_1, J_2, M_1 and M_2 on $\mathbb{D}$ proposed by Skorokhod in his seminal paper [38] immediately attracted much attention.

Among Skorokhod's topologies the finest, hence the closest to the uniform topology, was J_1. One says that that $x_n \in \mathbb{D}$ converges to $x_0 \in \mathbb{D}$ in J_1 topology if there exists a sequence of increasing homeomorphisms $\lambda_n : [0,1] \to [0,1]$, $(\lambda_n \in \Lambda)$ such that

$$(1) \qquad \sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0, \quad \sup_{t\in[0,1]} |x_n(\lambda_n(t)) - x_0(t)| \to 0, \text{ as } n \to \infty.$$

In the case when x_0 is continuous, the convergence in J_1 is equivalent to the uniform convergence. Further, if $x_n \to x_0$ in J_1 and x_0 admits a jump

$$\Delta x_0(t_0) = x_0(t_0) - x_0(t_0-) \neq 0$$

at some point $t_0 \in (0,1]$, then there exists a sequence $t_n \to t_0$ such that $\Delta x_n(t_n) \to \Delta x_0(t_0)$. Many natural and important functionals on $\mathbb{D}$, which are continuous in the uniform topology, are discontinuous in J_1. Fortunately, in most cases it is easy to describe the set of J_1-continuity points of such functionals. For example, if $\pi_t(x) = x(t)$ or $f_t(x) = \sup_{s\in[0,t]} x(s)$, then $\pi_t(\cdot)$ and $f_t(\cdot)$ are continuous at $x_0 \in \mathbb{D}$ if, and only if, x_0 is continuous at t. By a trivial modification of the definition, one obtains spaces $\mathbb{D}([0,T] : \mathcal{X})$, where $T > 0$ and $\mathcal{X}$ is a Polish space. Contrary to the case of continuous functions, the definition of the topology J_1 on $\mathbb{D}([0,+\infty) : \mathcal{X})$ requires some care and was given for $\mathcal{X} = \mathbb{R}^d$ by Lindvall [31] in 1973 only.

To cope with technically difficult topologies, Skorokhod [38] invented an approach being of independent interest.

1. The method of a single probability space

The method of a single probability space is the first important example of what is nowadays known as "coupling".

Theorem 1. *Let $(\mathcal{X}, \rho)$ be a Polish space (i.e. metric, separable and complete) and let $X_0, X_1, X_2, \ldots$ be random elements taking values in $\mathcal{X}$. Suppose the sequence $\{X_n\}$ converges in distribution to X_0:*

$$X_n \underset{\mathcal{D}}{\longrightarrow} X_0. \tag{2}$$

Then there exist $\mathcal{X}$-valued random elements $Y_0, Y_1, Y_2, \ldots$, defined on the unit interval $([0,1], \mathcal{B}_{[0,1]})$ equipped with the Lebesgue measure ℓ, such that

$$\text{the laws of } X_n \text{ and } Y_n \text{ coincide for } n = 0, 1, 2, \ldots, \tag{3}$$

$$\rho(Y_n(\omega), Y_0(\omega)) \to 0, \ \text{as } n \to \infty, \ \text{for each } \omega \in [0,1]. \tag{4}$$

The above device is often called "the almost sure Skorokhod representation", for in practice we need in (4) the convergence for ℓ-almost all ω only.

The almost sure Skorokhod representation can be used to trivialize proofs in the theory of convergence in distribution on Polish spaces (portmanteau theorem, convergence of moments, etc.), as it was done e.g. in [8]. The core applications, however, are related to functional convergence of stochastic processes (see the next section) and to convergence of empirical processes.

In the latter case Theorem 1 is not directly applicable, for the machinery of empirical processes operates in nonseparable metric spaces. A suitable "method of the single probability space" was introduced in the late sixtieth and early seventieth. In 1968 Dudley [12] generalized Skorokhod's idea to separable metric spaces. Then Wichura [45] (see also Fernandez [15]) constructed the Skorokhod representation in nonseparable metric spaces, provided the limiting law had separable range. In general, the price to be paid for the lack of completeness or separability was larger space required by the definition of the representation (typically a product space). Numerous examples of how the method works in the theory of empirical processes can be found in [44] (see also [13]).

There are many papers discussing various aspects of the a.s. Skorokhod representation in metric spaces. One can mention here [5], [29], [30], [42], [43].

A new trend in the theory was initiated in papers [9] and [17], where the following Balckwell-Dubbins-Fernique theorem was proved.

Theorem 2. *Let $(\mathcal{X}, \rho)$ be a Polish space. Then with every Borel probability measure μ on $\mathcal{X}$ one can associate a Borel mapping $Y_\mu : [0,1] \to \mathcal{X}$ in such a way that*

$$\ell \circ Y_\mu^{-1} = \mu, \tag{5}$$

and if a sequence $\{\mu_n\}$ weakly converges to μ_0, then

$$\rho(Y_{\mu_n}(\omega), Y_{\mu_0}(\omega)) \to 0, \ \text{as } n \to \infty, \ \text{for almost all } \omega \in [0,1]. \tag{6}$$

One can say that for Polish spaces there exists a Skorokhod parameterization of Borel probability measures.

It is interesting that for an *arbitrary* metric space $\mathcal{X}$, there exists a Skorokhod parameterization of the set of *tight* (or *Radon*) probability measures on $\mathcal{X}$. This result was proved in [10] by a new topological method, which reduces the problem to the case of a subset of interval $[0,1]$.

In the case of nonmetric spaces much less had been known for long time. Fernique [16], discussing convergence in distribution in Lusin spaces (in particular: in spaces of distributions like $\mathcal{S}'$ or $\mathcal{D}'$), exhibited an example of a sequence converging in distribution on a separable Hilbert space equipped with the weak topology, with no subsequence being uniformly tight (consequently: with no subsequence admitting the a.s. Skorokhod representation). Schief [37] provided an example of a Lusin space on which there is no Skorokhod parameterization of Radon probability measures. The first positive and important in applications result in this area was given by Jakubowski in [26].

Theorem 3. *Let* $(\mathcal{X},\tau)$ *be a topological space, for which there exists a countable family* $\{f_i : \mathcal{X} \to \mathbb{R}^1\}_{i\in\mathbb{I}}$ *of* τ*-continuous functions, separating points of* $\mathcal{X}$*. Then in every* uniformly tight *sequence* $X_1, X_2, \ldots$ *of* $\mathcal{X}$*-valued random elements one can find a subsequence* $\{X_{n_k}\}_{k\in\mathbb{N}}$ *and* $\mathcal{X}$*-valued random elements* $Y_0, Y_1, Y_2, \ldots$ *defined on* $([0,1], \mathcal{B}_{[0,1]}, \ell)$ *such that*

$$X_{n_k} \sim Y_k, \quad k = 1, 2, \ldots, \tag{7}$$

$$Y_k(\omega) \underset{\tau}{\longrightarrow} Y_0(\omega), \quad \text{as } k \to \infty,\ \omega \in [0,1]. \tag{8}$$

In particular, in assumptions of the above theorem, if $X_n \longrightarrow_{\mathcal{D}} X_0$ and $\{X_n\}$ is uniformly tight, then one obtains the a.s. Skorokhod representation *for subsequences*: in every subsequence $\{n_k\}$ one can find a further subsequence $\{n_{k_l}\}$ such that $\{X_{n_{k_l}}\}$ and X_0 admit the usual a.s. Skorokhod representation on $[0,1]$. There exist examples (see [10]) showing that on the space $\mathbb{R}_0^\infty$ of finite sequences equipped with the topology of the strict inductive limit, the a.s. Skorokhod representation for subsequences cannot be strengthened to the usual representation for the whole sequence.

For extensive discussion of the a.s. Skorokhod parameterization in both metric and nonmetric spaces we refer to [3].

2. Developments related to the topology J_1

Skorokhod applied his own approach in a series of papers on processes with independent increments [39], Markov processes [40] or stochastic differential equations [41]. Some results have been included into the famous textbook [19] and monograph [20], [21] and became commonly known.

It is interesting that Skorokhod's machinery worked perfectly despite the directly related to the topology J_1 metric

$$d(x,y) = \inf_{\lambda \in \Lambda} \big(\sup_{t\in[0,1]} |\lambda(t) - t| + \sup_{t\in[0,1]} |x(\lambda(t)) - y(t)|\big)$$

was *incomplete* on $\mathbb{D}$. What was important was the known and manageable form of conditionally compact subsets of $\mathbb{D}$ equipped with J_1. The same was also true for other Skorokhod's topologies. Paradoxically, at present the Skorokhod space with J_1 is considered as a classical illustration of the theory "tightness + identification of the limit" due to Prokhorov [35], designed for and valid mainly in Polish spaces. It was Kolmogorov [28] who showed that $\mathbb{D}$ with J_1 is topologically complete. The complete metric on $\mathbb{D}$ was given explicitly by Billingsley in [7].

The book by Billingsley [7], published in 1968, enormously stimulated the development of functional limit theorems in the seventieth and the eighties of the twentieth century. At the same time, a convenient criterion of tightness due to Aldous [1] (see also Rebolledo [36]) considerably reduced the efforts required for checking tightness for martingales, semimartingales, Markov processes, mixing sequences and so on. Description of main achievements of the period under discussion exceeds the frames of this short introduction. We refer to books by Ethier and Kurtz [14] and Jacod and Shiryaev [24] for exhaustive discussion of results of this golden era for limit theorems.

3. The Skorokhod topology for multiparameter processes

The Skorokhod space and the Skorokhod topology J_1 for processes indexed by elements of $[0,1]^d$ with $d > 1$, was constructed by Neuhaus [34] and Bickel and Wichura [6]. In this case the Skorokhod space consists of functions $x : [0,1]^d \to \mathcal{X}$ which are at each point right continuous (with respect to the natural partial order of $\mathbb{R}^d$) and admit limits in all "orthants". The convergence means uniform convergence relaxed by the "time" change independently on each coordinate. The history of research in this area shows that the multiparameter Skorokhod space is less natural than the usual $\mathbb{D}$.

Among possible generalizations one should mention here the ideas of Bass and Pyke [4] and the book by Ivanoff and Merzbach [23].

4. The Skorokhod topology on completely regular spaces

In the early 1980s, Kiyoshi Itô [22] obtained a stochastic process with values in the space of (Schwartz) distributions as a limit for suitably transformed measure-valued processes built upon a sequence of independent Brownian motions. Itô's followers - Mitoma [33], Fouque [18] and others - considered stochastic processes with values in the Skorokhod space $\mathbb{D}([0,1] : \mathcal{X})$, where $\mathcal{X}$ was $\mathcal{S}'$, $\mathcal{D}'$, *a nuclear space* or, more general *a completely regular space*. Such Skorokhod spaces was investigated in detail by Jakubowski [25]. A nice criterion of tightness obtained in [25] became a basic simplifying tool for research on superprocesses and other measure-valued processes intensively studied that time (see [11]).

5. Weak topologies on the Skorokhod Space

The fact that convergence in J_1 topology to a continuous limit is equivalent to the uniform convergence, implies that $\mathbb{C}([0,1])$ is a closed subset of $\mathbb{D}([0,1])$. In particular, a discontinuous element of $\mathbb{D}([0,1])$ cannot be approximated in J_1 by a sequence of continuous functions. Sometimes this is a drawback: if we deal with a smoothing procedure like convolution, it cannot be continuous in J_1. Other problems may be caused when jumps in the limit arise as local cumulations of small jumps, what is impossible in J_1 . Finally, in existence problems it is desirable to use as weak topology as possible, since it requires weak conditions for tightness. All these reasons bring interest also to the weaker Skorokhod's topologies J_2, M_1 and M_2. Among them practically only the topology M_1 proved to be useful. A typical example of limit theorem for the topology M_1, showing that this phenomenon is quite natural, is given in the paper by Avram and Taqqu [2].

One should mention that there exist also topologies on the Skorokhod space which are not Skorokhodian (see [32] and [27]). Both of them are weaker than the Skorokhod topologies and linear in the sense that the sum of convergent sequences converges to the sum.

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